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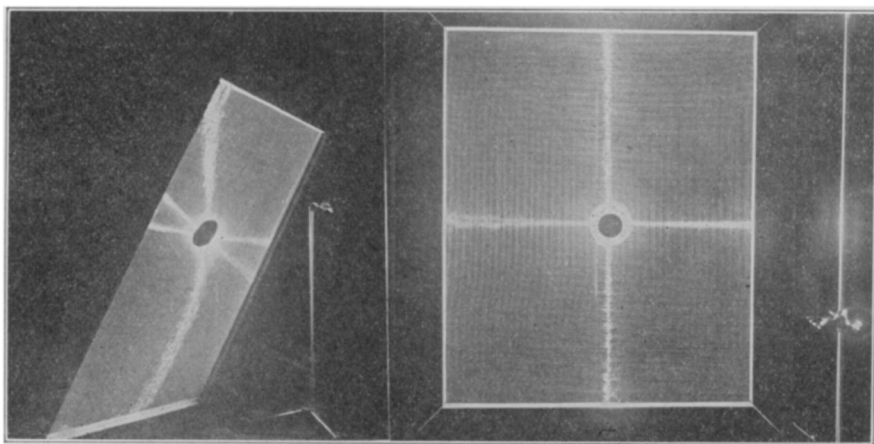
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of geometrical language in domains which might, at first, seem unsuited to such phraseology. If the order, m , of the matrices representing generalized numbers be allowed to become infinite in an appropriate manner, these generalized numbers become operators of the form used in the theory of integral equations. While it has not been customary to study integral equations in the domain of quaternions, some interesting results might be obtained in such a case.

LINES OF ILLUMINATION CAUSED BY THE PASSAGE OF LIGHT THROUGH A SCREEN.¹

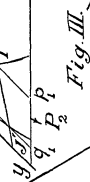
By WM. H. ROEVER, Washington University.

1. **Description.** On viewing a point source of light through a screen one will see on the screen several streamers of light emanating from the point in which the plane of the screen is pierced by the line connecting the eye with the light. The number of streamers which appear depends upon the direction of this line with respect to the plane of the screen. If, in particular, this line is perpendicular to the plane of the screen, there are four streamers, or two lines of light which cut at right angles. This is also the case for some other positions of the light and eye. In general, however, there are six streamers of light. On an old screen, these streamers are short and straight and one gets the impression of seeing a four- or six-pointed star. This is especially noticeable on looking at a source of light through the dust screen of a sleeping-car window. At first, when the source of light is far ahead of the train, the star will appear to have six points, but when the train is opposite the source of light the star appears to have only four points. If the screen is new and is made of polished wires of brass or copper, the streamers



extend to the frame and are well-defined curves. Views of this phenomenon are shown in the accompanying photographic reproductions. The black spot in these

¹ Presented to the American Mathematical Society, November 27, 1915. See *Bulletin of that Society*, vol. 22, p. 218.



reproductions is due to the presence of a small disk which was used to obscure the intense direct light from the source.

2. Cause. A close examination of the screen will show that the screening, from which it is made, must have been woven like cloth. The screening consists of a series of longitudinal wires, called the warp, which are interlaced with a series of transverse wires, called the woof. In the weaving process of raising certain of the longitudinal wires and depressing others so as to form a *shed* for the passage of the transverse wires, the warp wires become crinkly, or wavy, but the woof wires remain unbent. The result of this operation is, that after the screening is attached to the frame, the wires of the woof are practically straight, parallel to each other and lie in the plane of the screen (or rather between two parallel planes which are separated by a distance equal to the diameter of these wires). The wires of the warp, however, are crinkly, or wavy, because they bend in and out around the straight wires of the woof, and each such wire lies in a plane perpendicular to the wires of the woof (or rather, between two such planes which are separated by a distance equal to the diameter of this wire). A clear notion of this construction may be obtained from views 1 and 2 of Fig. I. In view 2 the wires of the woof are represented by their cross sections, which are small circles separated by intervals of constant length, and the wires of the warp are shown as broad wavy lines. In view 1, the wires of the woof are represented by broad unshaded straight lines, while those of the warp are represented by broad straight lines which are shaded so as to represent the bending which takes place in crossing the wires of the woof.

From this description it is clear that the bend points of the wires of the warp, like *M* and *N* in view 2 of Fig. I, either receive no light from the source, which is on one side of the screen, or are obscured from the observer who is on the other side. Therefore, of the wires of the warp, only those approximately straight portions, which lie between the bend points, are effective in reflecting the light which comes from a source on one side of the screen to an observer who is situated on the other side. The straight wires of the woof (or, rather, those portions not obscured by the wires of the warp) are also effective in reflecting this light.

As far as its ability to reflect light from a source on one side to an observer on the other side is concerned, the screen may be regarded as composed of three groups of short, needlelike, pieces of wire. Those of one group (the portions of the wires of the woof not obscured by, or in the shadow of, the wires of the warp) lie in the plane of the screen and are parallel to each other. Those of each of the other two groups (composed of the short straight portions of the wires of the warp) are parallel to one another, are inclined to the plane of the screen at a constant angle and are perpendicular to the wires of the woof. View 2, Fig. I, represents the wires of one of these two groups as parallel to the line *i* and those of the other as parallel to the line *j*.

The wires of the screen, or rather the groups of short portions just described, reflect the light which falls upon them from the source, and many of the reflected rays reach the eye of the observer (even though it be on the side of the screen

opposite to that of the light). Those points of the wires where the rays are reflected which reach the eye of the observer, are the images of the source of light in the reflecting surface of the wires, and are called brilliant points. On account of the fineness of the mesh of the screen, consecutive brilliant points are close together, and it is the aggregate of these closely packed brilliant points which forms the streamers of light observed.¹

3. Assumptions, Definitions, Tests. The solution of our problem may be much simplified without any appreciable loss in value, by assuming that the wires composing the screen are mathematical curves. For, the position of a brilliant point of a wire of small cross section differs very slightly from the position of the point which is approached by this brilliant point, as the cross section of the wire is diminished indefinitely. The geometric description of this limit point on the mathematical curve approached by the wire furnishes a definition for the brilliant point of a curve.

From the laws of the reflection of light it follows that the brilliant point of a surface may be described according to

DEFINITION 1. *A point P is said to be a brilliant point of a surface t with respect to a source of light P_1 and an observer's eye P_2 , if the internal bisector of the angle P_1PP_2 is the normal to this surface at the point P . See Fig. II.*

If now the reflecting surface is supposed to be tubular² (like that of a wire) and the diameter of a cross section of this tube be regarded as infinitesimal (*i.e.*, as approaching zero), the brilliant point approaches a point on the mathematical curve approached by the tube which may be described by

DEFINITION 2. *A point P is said to be a brilliant point of a curve c with respect to a source of light P_1 and an observer's eye P_2 , if the internal bisector of the angle P_1PP_2 is a normal to this curve at the point P . See Fig. II.*

It is of course evident that the rays PP_1 and PP_2 do not, in general, lie in the same tangent plane to the curve at the point P . If in definitions 1 and 2 the words "internal bisector" be replaced by the words "external bisector," these definitions become respectively those of the virtual brilliant point of a surface and a curve. Such points are not visible, but they present themselves in the analytic treatment when certain equations are freed from radicals.³

The method of treatment used in this paper was selected not only for the purpose of obtaining the equation of the brilliant point locus in question, but also

¹ At the suggestion of a referee, attention is here called to the fact that the spacing of the wires of the screen is so great as to make the phenomenon of diffraction negligible. The phenomenon described in this paper should therefore not be confused with the phenomenon observed in looking at a source of light through the stretched cover of an open umbrella.

² A tubular surface may be defined as the envelope of a sphere of constant radius which moves so that its center always lies on a curve (plane or skew), which is called the *axis*. If, in particular, the axis is a straight line, the surface becomes a cylinder of revolution; and if the axis is a circle, the surface is a torus.

³ In definitions given by the author in the *Transactions of the American Mathematical Society*, vol. 9, pp. 245-279, the term *actual brilliant point* was used to designate the point which is here called simply *brilliant point*, and the term *brilliant point* was used to include both actual and virtual brilliant points.

with a view to furnishing a method of constructing this locus by means of circles and straight lines.¹

In order to understand this method it will be necessary to know the following property of a sphere. If any point P on the surface of a sphere be connected by right line segments with each of a pair of points P_1 and P_2 , which are conjugate relative to an inversion with respect to this sphere, and also with the points Q and R in which the diameter containing P_1 and P_2 pierces the sphere, the latter segments bisect internally and externally the angle formed by the former.

Any two points P_1 and P_2 of space constitute such a pair of points for any member of a certain one-parameter family of spheres. If r_1 and r_2 denote the distances of a general point P of space from the points P_1 and P_2 , respectively, this family of spheres is represented by the equation

$$\frac{r_1}{r_2} = k, \quad (1)$$

in which k is a (positive) constant for any particular sphere of the family.

This property of the sphere furnishes the following

TEST. *To determine whether a point P of a curve C is a brilliant point of C with respect to two points P_1 and P_2 , construct the normal plane to the curve at the point P and denote by Q the point in which this normal plane cuts the line P_1P_2 . If Q lies between P_1 and P_2 , pass through it a sphere having P_1 and P_2 as conjugate points. Then the point P is, or is not, a brilliant point of the curve C with respect to P_1 and P_2 , according as it does, or does not, lie on the surface of this sphere. If, in particular, the normal plane contains both P_1 and P_2 , the point P is a brilliant point.²*

The truth of the statement of this test lies in the fact that the normal line PQ to the curve C at the point P does, or does not, bisect internally the angle P_1PP_2 according as P does, or does not, lie on the sphere which passes through Q and has P_1 and P_2 as a pair of conjugate points. If, in particular, the normal plane contains both P_1 and P_2 , the bisector of the angle P_1PP_2 is surely a normal to the curve at the point P .

4. Solution of the Problem. In order to find the equations of the streamers of light described in section 1, we will make use of the test just given. To this end we will choose (as in Fig. III) a set of rectangular axes, Ox , Oy , Oz , of which the origin O is the point where the line P_1P_2 (connecting the eye P_2 with the source of light P_1) pierces the plane of the screen, which is taken to be coincident with the plane xOy , and the axis Oy is coincident with the wire of the woof passing through the origin O . We will further choose on the line P_1P_2 an axis $O\zeta$ of which the origin is the point O and the positive sense is from P_2 (taken below

¹ For other methods of treatment the reader is referred to an article by the author entitled "Brilliant point phenomena" published in the *Washington University Studies*, vol. 8, Scientific Series, pp. 131-160, 1921.

² To get the corresponding test for a virtual brilliant point, it is only necessary to replace the words "lies between P_1 and P_2 " by the words "lies beyond the limits of the segment P_1P_2 ." If, in particular, the normal plane is parallel to the line P_1P_2 the sphere becomes the perpendicular bisecting plane of the segment P_1P_2 .

the screen) to P_1 (above the screen), and we will denote by ρ the (perpendicular) distance of a general point P from this axis. We will denote by A the point halfway between P_1 and P_2 , and by h and c the distances OA and AP_1 , respectively.

The quantities r_1 and r_2 of equation (1) may now be expressed by the formulas $r_1 = \sqrt{[(\zeta - h - c)^2 + \rho^2]}$, $r_2 = \sqrt{[(\zeta - h + c)^2 + \rho^2]}$ and the equation (1) thus becomes

$$\frac{(\zeta - h)^2 + \rho^2 + c^2}{\zeta - h} = 2l, \quad (2)$$

in which $l = c(1 + k^2)/(1 - k^2)$ is the distance of the center of this sphere from the point A .

To express the coördinates ρ and ζ in terms of x, y, z , we will denote by α, β, γ the direction angles of the axis $O\zeta$ with respect to the set of axes, Ox, Oy, Oz . Then, evidently,

$$\begin{aligned} \rho^2 + \zeta^2 &= x^2 + y^2 + z^2, \\ \zeta &= x \cos \alpha + y \cos \beta + z \cos \gamma. \end{aligned} \quad (3)$$

Making these substitutions, equation (2) becomes

$$\frac{x^2 + y^2 + z^2 - 2h(x \cos \alpha + y \cos \beta + z \cos \gamma) + h^2 + c^2}{x \cos \alpha + y \cos \beta + z \cos \gamma - h} = 2l. \quad (4)$$

By allowing l in this equation to take on all real values, we obtain all the spheres which have P_1 and P_2 as a pair of conjugate points.

We will denote by $\bar{Q}$ (Fig. III) the point in which the line P_1P_2 (*i.e.*, the axis $O\zeta$) is cut by a common normal plane of the wires of the woof. This plane has the equation

$$y = \bar{y} = \text{constant} (= Om_0 \text{ in Fig. III}), \quad (5)$$

and therefore $\bar{Q}$ is determined by the relations $\bar{\zeta} = O\bar{Q} = \bar{y}/\cos \beta$, $\bar{\rho} = 0$. Substituting these values for ζ and ρ in equation (2) we obtain the expression

$$2l = \left[\frac{\bar{y}^2}{\cos^2 \beta} - 2h \frac{\bar{y}}{\cos \beta} + h^2 + c^2 \right] / \left[\frac{\bar{y}}{\cos \beta} - h \right], \quad (6)$$

which gives the value of l for that sphere of the family (4) which passes through $\bar{Q}$. This sphere cuts the plane $zO\zeta$ in the circle $\bar{Q}S$ of Fig. III and the plane (xOy) of the screen in the circle $L'SI'J'$. The latter circle is represented by the equation which is obtained from equation (4) by giving $2l$ the value (6) and putting $z = 0$, *i.e.*, by the equation

$$\frac{x^2 + y^2 - 2h(x \cos \alpha + y \cos \beta) + h^2 + c^2}{x \cos \alpha + y \cos \beta - h} = \frac{\frac{\bar{y}^2}{\cos^2 \beta} - 2h \frac{\bar{y}}{\cos \beta} + h^2 + c^2}{\frac{\bar{y}}{\cos \beta} - h}. \quad (7)$$

On the other hand, the plane (5) cuts the plane (xOy) of the screen in the line m_0m_1 (Fig. III), which is represented by the equations

$$y = \bar{y}, \quad z = 0. \quad (8)$$

According to the Test (stated in the preceding section), those points only (like L' of Fig. III) of the line (8) [m_0m_1 of Fig. III] are brilliant points of wires of the woof, with respect to P_1 and P_2 , in which this line is cut by the circle (7) [$L'SI'J'$ of Fig. III]. Consequently, the locus of brilliant points (actual and virtual) of the wires of the woof is obtained by giving $\bar{y}$ ($= Om_0$ in Fig. III) all possible real values, *i.e.*, by eliminating $\bar{y}$ between equations (7) and (8). The resultant of this elimination may be written in the form:

$$xy^2 \cos \alpha - x^2y \cos \beta + h(x^2 \cos^2 \beta - y^2 \sin^2 \beta)$$

$$+ (c^2 - h^2) \cos \beta (x \cos \alpha \cos \beta - y \sin^2 \beta) = 0. \quad (9)$$

In order to determine the locus of brilliant points of the short (approximately) straight portions of the wires of the warp, we will denote by Q (Fig. III) the point in which the line P_1P_2 (*i.e.*, the axis $O\zeta$) is cut by a plane perpendicular to one, or other, of the two groups of such short portions. The equation of such a plane may be written in the form

$$\pm \epsilon z + x = \bar{x} = \left\{ \begin{array}{l} Op_0 \\ Oq_0 \end{array} \text{ in Fig. III } \right\}, \quad (10)$$

in which $\bar{x}$ is the distance of the horizontal trace of this plane from the axis Oy , and $\epsilon = \tan \phi$ where ϕ is the angle of inclination of the short straight portions of the wires of the warp to the plane (xOy) of the screen. Then, for the upper sign equation (10) represents a plane perpendicular to the line i of view 2, Fig. I, and for the lower sign it represents a plane perpendicular to the line j . Let us denote the coördinates of $\tilde{Q}$ by $(\tilde{x}, \tilde{y}, \tilde{z})$. Since $\tilde{Q}$ is a point of the plane (10), we have $(\pm \epsilon \tilde{z} + \tilde{x})/\tilde{\zeta} = \bar{x}/\tilde{\zeta}$ or $\pm \epsilon \cos \gamma + \cos \alpha = \bar{x}/\tilde{\zeta}$, whence $\tilde{\zeta} = O\tilde{Q} = \bar{x}/e$, $\tilde{\rho} = 0$, where

$$e = \cos \alpha \pm \epsilon \cos \gamma. \quad (11)$$

Putting these values for ζ and ρ in equation (2), we obtain the relation

$$2l = \left[\frac{\bar{x}^2}{e^2} - 2h \frac{\bar{x}}{e} + h^2 + c^2 \right] / \left[\frac{\bar{x}}{e} - h \right], \quad (12)$$

which gives the value of l for that sphere of the family (4) which passes through $\tilde{Q}$. This sphere cuts the plane ($z = 0$) of the screen in the circle

$$\frac{x^2 + y^2 - 2h(x \cos \alpha + y \cos \beta) + h^2 + c^2}{x \cos \alpha + y \cos \beta - h} = \frac{\frac{\bar{x}^2}{e^2} - 2h \frac{\bar{x}}{e} + h^2 + c^2}{\frac{\bar{x}}{e} - h}. \quad (13)$$

It is no loss of generality to assume, as in Fig. III, that $\tilde{Q}$ coincides with $\bar{Q}$. Hence equation (13) is also represented by the circle $L'SI'J'$ in Fig. III. The plane (10) cuts the plane of the screen in the trace whose equations are

$$x = \bar{x}, \quad z = 0. \quad (14)$$

In Fig. III this trace is represented by the line p_0p_1 or q_0q_1 according as the upper or lower sign of equation (10) is considered. According to the Test, those points only (like I' or J' of Fig. III) of the line (14) are brilliant points, with respect to P_1 and P_2 , of wires of the warp (or rather, of the short straight portions thereof) in which this line is cut by the circle (13). Consequently, the locus of brilliant points (actual and virtual) of the short straight portions of the wires of the warp (and hence of the wires of the warp themselves, since, as we have seen, the bend points yield no brilliant points) may be obtained by giving $\bar{x}$ all possible real values, that is, by eliminating $\bar{x}$ between equations (13) and (14). The eliminant thus obtained may be written in the form

$$\pm \epsilon x^3 \cos \gamma - x^2 y \cos \beta + \epsilon xy^2 + h[(1 - e^2)x^2 - e^2y^2] \\ + e(c^2 - h^2)[(1 - e \cos \alpha)x - ey \cos \beta] = 0, \quad (15)$$

in which e stands for the expression (11). For the upper sign this equation represents the brilliant point locus for the short straight portions of the wires of the warp which are parallel to the line i of view 2, Fig. I, and for the lower sign it represents the corresponding locus for the portions parallel to the line j .

The space construction represented in (cavalier) perspective in Fig. III is executed in the drawing plane of Fig. I by means of the Mongean method of descriptive geometry; thus the graphs of the curves (9) and (15) are obtained in Fig. I. The curve $OL/L_v/L_v''$ is the graph of the equation (9), the curve $OJ'J_v''/J_v''$ is the graph of equation (15) for the lower sign, and the curve $O'I'$ is the graph of equation (15) for the upper sign. The full portions of these curves are the loci of the (actual) visible brilliant points and the dashed portions are the loci of the invisible virtual brilliant points.

If we equate to zero the first degree terms of equations (9) and (15) we obtain the equations

$$(1 - \cos^2 \beta)y - \cos \alpha \cos \beta \cdot x = 0, \quad (16)$$

$$[1 - \cos \alpha(\cos \alpha \pm \epsilon \cos \gamma)]x - \cos \beta(\cos \alpha \pm \epsilon \cos \gamma)y = 0, \quad (17)$$

of the tangents to these curves at the origin. These are, in fact, the equations of the short straight streamers which we observed in viewing a light through a tarnished screen. From equation (17) it is evident that the two lines which this equation represents coalesce when $\beta = 90^\circ$ (*i.e.*, when the line connecting the eye with the light is perpendicular to the wires of the woof). In this case equations (16) and (17) become $y = 0$ and $x = 0$ respectively. Thus for $\beta = 90^\circ$ the six-pointed star becomes a four-pointed star.